ELSEVIER Journal of Pure and Applied Algebra 117 & 118 (1997) 3–21

JOURNAL OF PURE AND APPLIED ALGEBRA

The number of outside corners of monomial ideals

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The outside corners of a monomial ideal are the maximal standard monomials modulo that ideal. Let $c_n(p)$ be the maximal number of outside corners of any monomial ideal generated by p monomials in n variables. We show that $c_n(p) = \Theta(p^{[n/2]})$ for fixed n. An exact calculation for $n = 4$ shows that the function $c_n(p)$ is not a polynomial in p. © 1997 Elsevier Science B.V.

1991 Math. Subj. Class.: 05, 13, 16

1. Introduction

Let $c_n(p)$ denote the maximal number of maximally (w.r.t. divisibility) standard monomials for a monomial ideal $(W_1, \dots, W_p)$ in $R = k[X_1, \dots, X_n]$. This quantity is of interest for Computational Commutative Algebra since it implies an effective version of Macaulay's result that every ideal I in R can be "co-generated" by finitely many functionals $R \to k$, meaning that I is the largest ideal contained in the kernels of finitely many functionals $R \to k$ [4, pp. 69 and 91]. In the author's Ph.D. thesis [1] it is shown that if I has a Gröbner basis consisting of p elements then I can be co-generated by $c_n(p)+1$ functionals $R \to k$. This bound is sharp.

In this paper we study $c_n(p)$ and estimate its behaviour for fixed n and varying p. We establish that for $n < 4$ and $p \geq n$, we have $c_n(p) = (n-1)(p-n)+1$ and, in general, $c_n(p) = \Theta(p^{[n/2]})$ for fixed n. We also show that $c_4(p) = (p^2 - 3p - 2)/2$ for $4 \leq p \leq 12$.

There seems to be a relation to the Upper Bound Theorem (see [5, Theorem 8.23, p. 254]) which concerns the maximal number $f_{n-1}(p)$ of $(n-1)$-faces of an n-dimensional polytope with p vertices. The above results suggest that $c_n(p) = f_{n-1}(p) - 1$ might hold. However, we will show that $c_4(13) = 63 < (13^2 - 3 \cdot 13 - 2)/2 =$

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PII S0022-4049(97)00002-9

$f_3(13)-1$, thereby invalidating this wishful relationship. We also conclude that, unlike $f_{n-1}(p)$, $c_n(p)$ is not a polynomial in p.

We begin by introducing terminology and definitions. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0$ the set of nonnegative integers. The set $\mathbb{N}_0^n$ is a partially ordered set (poset) under the component wise order.

Definition 1. A subset $U \subseteq \mathbb{N}_0^n$ is a *filter* if $x \geq u \in U \Rightarrow x \in U$ for all $x, u \in \mathbb{N}_0^n$. Likewise $D \subseteq \mathbb{N}_0^n$ is a *co-filter* if $x \leq d \in D \Rightarrow x \in D$. For a filter U of $\mathbb{N}_0^n$ define $\mathrm{Out}(U)$ to be the maximal elements of $\mathbb{N}_0^n \setminus U$. The elements of $\mathrm{Out}(U)$ are called *outer corner points* or *corner points* of U.

Remarks. Clearly for $x \in \mathrm{Out}(U)$ all $x' > x$ are in U, so one can picture x as a point which is "outside" the filter but lies in a "corner" formed by the members of the filter. This is why we use the word "out" for $\mathrm{Out}(U)$ and we call each $x \in \mathrm{Out}(U)$ an "outer corner point". Note also:

- U is a filter of $\mathbb{N}_0^n$ if and only if $U + \mathbb{N}_0^n \subseteq U$.
- For a filter U of $\mathbb{N}_0^n$, $\mathrm{Out}(U)$ consists precisely of those points $\tilde{x} \in \mathbb{N}_0^n$ satisfying $\tilde{x} \notin U$ and $\tilde{x} + \tilde{e}_i \in U$ for all $i \in \{1, \ldots, n\}$. This is the characterization of Out we will mostly use.
- For $\tilde{x} \in \mathbb{N}_0^n$ let $C(\tilde{x})$ be the filter *generated* by $\tilde{x}$, that is the orthant $\tilde{x} + \mathbb{N}_0^n$. A subset $G \subseteq U$ *generates* U if and only if $U = \bigcup_{\tilde{x} \in G} C(\tilde{x})$.

Recall the familiar combinatorial lemma of Dickson's (see [3, pp. 163, 189]):

Lemma 2 (Dickson). *Every filter of $\mathbb{N}_0^n$ is a finite union of orthants. Hence every filter of $\mathbb{N}_0^n$ has only finitely many minimal elements.*

Definition 3. Let $\mathscr{C}_p^n$ denote the set of filters of $\mathbb{N}_0^n$ which can be written as a union of p or fewer orthants. Define $c_n(p) \in \{0, 1, \ldots\} \cup \{\infty\}$ as

$$c_n(p) = \max_{U \in \mathscr{C}_p^n} |\mathrm{Out}(U)|. \tag{1}$$

Looking at Fig. 1, we see a filter of $\mathbb{N}_0^2$ generated by 5 elements (black dots) together with its 4 outer corner points (circles with dots in them).

Denote the filter generated by $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^n$ by $\langle \tilde{m}_1, \ldots, \tilde{m}_p \rangle$.

Lemma 4. *For $n < 4$ and $n \leq p$ we have $c_n(p) \geq (n-1)(p-n)+1$.*

Proof. By Definition 1, we have for $n=1$ that $\{0\} \subseteq \mathrm{Out}(\langle 1, 2, \ldots, p \rangle)$ and for $n=2$ that $\{(p-2,0), (p-3,1), \ldots, (0, p-2)\} \subseteq \mathrm{Out}(\langle (p-1,0), (p-2,1), \ldots, (0, p-1) \rangle)$, so for $n < 3$ and $n \leq p$ we have by Definition 3 $c_n(p) \geq (n-1)(p-n)+1$.

For $n=3$ we see that $\{(0,0,0)\} \subseteq \mathrm{Out}(\langle (1,0,0), (0,1,0), (0,0,1) \rangle)$ and hence $c_3(3) \geq 1$.

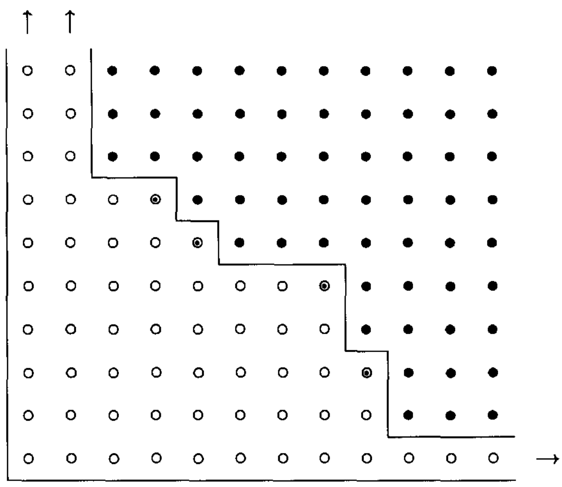

Fig. 1. An example of $U \in \mathscr{C}_5^2$.

Secondly we can see that for $m \geq 1$ the set $\mathrm{Out}(\langle(0, 0, 2m+1), (0, 2m+1, 0), (2i+1, m-i, m-i) : i = 1, 2, \ldots, m\rangle)$ contains the $2m+1$ elements $(0, 2m, 2m)$, $(2i, 2m, m-i), (2j, m-j, 2m) : i, j = 1, 2, \ldots, m$ of $\mathbb{N}_0^3$, and therefore $c_3(m+3) \geq 2m+1$. Hence $c_3(p) \geq 2p-5$ for $p \geq 3$. $\square$

Proposition 5. *For integers $n, m, p, q > 0$ we have $c_{n+m}(p+q) \geq c_n(p)c_m(q)$.*

Proof. Let $\tilde{u}_1, \ldots, \tilde{u}_p, \tilde{r}_1, \ldots, \tilde{r}_K \in \mathbb{N}_0^n$ be such that $\{\tilde{r}_1, \ldots, \tilde{r}_K\}$ is contained in $\mathrm{Out}(\langle\tilde{u}_1, \ldots, \tilde{u}_p\rangle)$, and let $\tilde{v}_1, \ldots, \tilde{v}_q, \tilde{t}_1, \ldots, \tilde{t}_L \in \mathbb{N}_0^m$ be such that $\{\tilde{t}_1, \ldots, \tilde{t}_L\} \subseteq \mathrm{Out}(\langle\tilde{v}_1, \ldots, \tilde{v}_q\rangle)$.

The set $\{(\tilde{r}_i, \tilde{t}_j) : 1 \leq i \leq K,\ 1 \leq j \leq L\} \subseteq \mathbb{N}_0^n \times \mathbb{N}_0^m$ is now a subset of $\mathrm{Out}(\langle(\tilde{u}_1, \tilde{0}), \ldots (\tilde{u}_p, \tilde{0}), (\tilde{0}, \tilde{v}_1), \ldots, (\tilde{0}, \tilde{v}_q)\rangle) \subseteq \mathbb{N}_0^n \times \mathbb{N}_0^m = \mathbb{N}_0^{n+m}$. From this we see that if $c_n(p) \geq K$ and $c_m(q) \geq L$ then $c_{n+m}(p+q) \geq K \cdot L$ holds. Hence $c_{n+m}(p+q) \geq c_n(p)c_m(q)$. $\square$

Recall that for a real number x, $[x]$ denotes the largest integer $\leq x$.

Lemma 6. *For $n \geq 1$ and $p \geq 3$ we have $c_n([n/2](p+1)) \geq p^{[n/2]}$.*

Proof. For $n = 2l$ we get by Lemma 4 and Proposition 5 $c_{2l}(l(p+1)) \geq c_2(p+1)^l \geq p^l$ for all $p \geq 1$. For $n = 2l+1$ we get in the same way $c_{2l+1}(l(p+1)) \geq c_2(p+1)^{l-1}c_3(p+1) \geq p^{l-1}(2p-3) \geq p^l$ for all $p \geq 3$. $\square$

Remark. If we fix $n > 1$ and look at $c_n(p)$ as a function of one variable p, then clearly $c_n(\cdot)$ is an increasing function by definition. Lemma 6 can be used to show that for a fixed n the function $c_n(p)/p^{[n/2]}$ is bounded below by a positive integer (see summarized results in Theorem 19).

2. An upper bound for $c_n(p)$

Definition 7. For a filter U in $\mathbb{N}_0^n$, we say that $\tilde{x} \in \mathbb{N}_0^n$ *is i-adjacent to* U if for some $i \in \{1,\dots,n\}$ we have $\tilde{x} \notin U$ and $\tilde{x}+\tilde{e}_i \in U$. We will say that $\tilde{x}$ is *adjacent to* U if there is an $i \in \{1,\dots,n\}$ such that $\tilde{x}$ is i-adjacent to U.

Remark 8. Let $\tilde{m} \in \mathbb{N}_0^n$. If $\tilde{x} \in \mathbb{N}_0^n$ is i-adjacent to $C(\tilde{m})$, then $x_i + 1 = m_i$.

If now $\tilde{x}, \tilde{m}_1, \dots, \tilde{m}_n \in \mathbb{N}_0^n$ are such that $\tilde{x}$ is i-adjacent to $C(\tilde{m}_i)$ for all i, then the previous remark asserts that $\tilde{x} = (m_{11}-1, \dots, m_{nn}-1)$ and that $\tilde{m}_1, \dots, \tilde{m}_n$ are all distinct. With this in mind, we can easily get the following lemma.

Lemma 9. *For $n \geq 1$ we have $c_n(n) = 1$ and $c_n(p) = 0$ for $p < n$.*

Remark. It follows from the definition of Out that $\mathrm{Out}(U)$ is contained in the union of the sets $\mathrm{Out}(V)$, where V ranges over the filters generated by all n-element subsets of $\{\tilde{m}_1, \dots, \tilde{m}_p\}$. Since each of these sets $\mathrm{Out}(V)$ has cardinality at most 1, $\mathrm{Out}(U)$ has cardinality at most $p!/n!(p-n)!$. A better bound will be found later in this section.

For a fixed $n \in \mathbb{N}$ define the map $P_i : \mathbb{N}_0^n \to \mathbb{N}_0$ simply as $P_i(\tilde{x}) = x_i$, the ith component, and let $\pi_{i_1 \dots i_k} : \mathbb{N}_0^n \to \mathbb{N}_0^{n-k}$ be the projection where components $i_1, \dots, i_k$ are omitted. In a context where n is given, π will *throughout this paper* mean the map $\pi_n : \mathbb{N}_0^n \to \mathbb{N}_0^{n-1}$.

Lemma 10. *For any filter $U \subseteq \mathbb{N}_0^n$ we have the following*:
(i) $|\mathrm{Out}(U)| = |\pi_i(\mathrm{Out}(U))|$ *for all* $i \in \{1,\dots,n\}$.
(ii) $\mathrm{Out}(\pi(U)) \cap \pi(\mathrm{Out}(U)) = \emptyset$.
(iii) $|\mathrm{Out}(U)| + |\mathrm{Out}(\pi(U))| = |\pi(\mathrm{Out}(U)) \cup \mathrm{Out}(\pi(U))|$.

For the remainder of this paper, whenever the symbol U occurs, it will denote a filter generated by a p-tuple of elements written $\tilde{m}_1, \dots, \tilde{m}_p$; that is $U = \bigcup_{i=1}^{p} C(\tilde{m}_i) \subseteq \mathbb{N}_0^n$. Thus, whenever we specify elements $\tilde{m}_1, \dots, \tilde{m}_p$, this will be understood to specify U. Also U^- will always mean $\bigcup_{i=1}^{p-1} C(\tilde{m}_i)$. If $p+1$ points $\tilde{m}_1, \dots, \tilde{m}_{p+1}$ are given, then U will mean $\bigcup_{i=1}^{p} C(\tilde{m}_i)$, and U^+ will mean $\bigcup_{i=1}^{p+1} C(\tilde{m}_i)$. If we for some reason put primes or stars on the points $\tilde{m}_1, \dots, \tilde{m}_p$, say $\tilde{m}'_1, \dots, \tilde{m}'_p$ or $\tilde{m}^*_1, \dots, \tilde{m}^*_p$, then U' and U^* will always mean $\bigcup_{i=1}^{p} C(\tilde{m}'_i)$ and $\bigcup_{i=1}^{p} C(\tilde{m}^*_i)$ respectively. All this can be combined; for example, U^{*+} will mean $\bigcup_{i=1}^{p+1} C(\tilde{m}^*_i)$.

Lemma 11. *If $\tilde{m}_1, \dots, \tilde{m}_p \in \mathbb{N}_0^n$ are such that $m_{pn} = \max_i\{m_{in}\}$ then*
(i) $\mathrm{Out}(U^-) \subseteq \mathrm{Out}(U)$.
(ii) $\pi(\mathrm{Out}(U) \setminus \mathrm{Out}(U^-)) \subseteq \mathrm{Out}(\pi(U^-))$.
(iii) $|\mathrm{Out}(U)| \leq |\mathrm{Out}(U^-)| + |\mathrm{Out}(\pi(U^-))|$.

Proof. To prove the first statement, let $\tilde{x} \in \mathrm{Out}(U^-)$. There is an $i \in \{1, \ldots, p-1\}$ such that $\tilde{x}$ is n-adjacent to $C(\tilde{m}_i)$. By Remark 8 $x_n + 1 = m_{in} \le m_{pn}$, so $\tilde{x} \notin C(\tilde{m}_p)$ and therefore $\tilde{x} \notin U$. Since $U^- \subseteq U$ we have $\tilde{x} \in \mathrm{Out}(U)$.

To prove the second statement let $\tilde{x} \in \mathrm{Out}(U)$. We shall first show that $\tilde{x} + \tilde{e}_l \in U^-$ for all $l < n$, then deduce from this that if $\tilde{x} \notin \mathrm{Out}(U^-)$, we must have $\pi(\tilde{x}) \in \mathrm{Out}(\pi(U^-))$. There is an $i \in \{1, \ldots, p\}$ such that $\tilde{x}$ is n-adjacent to $C(\tilde{m}_i)$. By Remark 8, we have $x_n + 1 = m_{in}$ and, hence, $m_{pn} > x_n$. If $l \in \{1, \ldots, n\}$ and $\tilde{x} + \tilde{e}_l \in C(\tilde{m}_p)$, then we get $P_n(\tilde{x} + \tilde{e}_l) \ge P_n(\tilde{m}_p) = m_{pn} > x_n = P_n(\tilde{x})$ and, hence, $l = n$ must hold. So for $\tilde{x} \in \mathrm{Out}(U)$, we have $\tilde{x} \notin U^-$ and

$$\tilde{x} + \tilde{e}_l \in U^- \quad \text{for all } l \in \{1, \ldots, n-1\}. \tag{2}$$

If in addition we assume $\tilde{x} \notin \mathrm{Out}(U^-)$, then we have

$$\tilde{x} + \tilde{e}_n \in C(\tilde{m}_p) \backslash U^-, \tag{3}$$

since, otherwise, it would be in $\mathrm{Out}(U^-)$ by (2). Now because $\tilde{x} \notin C(\tilde{m}_p)$, we get $x_n + 1 = m_{pn}$ by Remark 8 and this holds for all $\tilde{x} \in \mathrm{Out}(U) \backslash \mathrm{Out}(U^-)$.

Let $\tilde{x} \in \mathrm{Out}(U) \backslash \mathrm{Out}(U^-)$ and assume $\pi(\tilde{x}) \in \pi(U^-)$. This means there is $y \in \mathbb{N}_0$ such that $(\pi(\tilde{x}), y) \in U^-$. Since $\tilde{x} = (\pi(\tilde{x}), x_n) \notin U^-$, and U^- is a filter, $y > x_n$ and hence there is a smallest integer $z \in \{x_n + 1, \ldots, y\}$ such that $(\pi(\tilde{x}), z) \in U^-$. Hence, there is an $i \in \{1, \ldots, p-1\}$ such that $(\pi(\tilde{x}), z-1)$ is n-adjacent to $C(\tilde{m}_i)$. By Remark 8 we have $z = m_{in}$. By definition of z and our choice of m_{pn}, we now have $m_{in} = z \ge x_n + 1 = m_{pn} \ge m_{in}$ and hence $z = x_n + 1$. This implies $\tilde{x} + \tilde{e}_n = (\pi(\tilde{x}), z) \in U^-$, which cannot be by (3). We therefore have for $\tilde{x} \in \mathrm{Out}(U) \backslash \mathrm{Out}(U^-)$ that $\pi(\tilde{x}) \notin \pi(U^-)$, and by (2), we get $\pi(\tilde{x}) + \tilde{e}_l \in \pi(U^-)$ for all $l \in \{1, \ldots, n-1\}$. Therefore, $\pi(\tilde{x}) \in \mathrm{Out}(\pi(U^-))$.

Finally, to prove the third and the last statement, we have by Lemma 10 and the second statement that

$$\begin{aligned} |\mathrm{Out}(U)| &= |\mathrm{Out}(U) \backslash \mathrm{Out}(U^-)| + |\mathrm{Out}(U) \cap \mathrm{Out}(U^-)| \\ &\le |\pi(\mathrm{Out}(U) \backslash \mathrm{Out}(U^-))| + |\mathrm{Out}(U^-)| \\ &\le |\mathrm{Out}(\pi(U^-))| + |\mathrm{Out}(U^-)|. \qquad \square \end{aligned}$$

Lemma 12. *If $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^n$ are such that $m_{pn} > m_{in}$ for all $i < p$, then $\pi(\mathrm{Out}(U)) \cup \mathrm{Out}(\pi(U))$ is the disjoint union of the sets $\pi(\mathrm{Out}(U^-)) \cup \mathrm{Out}(\pi(U^-))$ and $\mathrm{Out}(\pi(U)) \backslash \mathrm{Out}(\pi(U^-))$.*

Proof. Let

$$A = \pi(\mathrm{Out}(U)) \cup \mathrm{Out}(\pi(U)),$$

$$B = \pi(\mathrm{Out}(U^-)) \cup \mathrm{Out}(\pi(U^-)),$$

$$C = \mathrm{Out}(\pi(U)) \backslash \mathrm{Out}(\pi(U^-)).$$

First let us show $B \cap C = \emptyset$. Of the two terms whose union defines B, the first one is by Lemma 11 contained in $\pi(\mathrm{Out}(U))$, which by Lemma 10 is disjoint from $\mathrm{Out}(\pi(U))$ and hence disjoint from C, while the second is disjoint from C by definition of the latter set.

Let us show now that $A \subseteq B \cup C$. Clearly $\mathrm{Out}(\pi(U)) \subseteq [\mathrm{Out}(\pi(U)) \backslash \mathrm{Out}(\pi(U^-))] \cup \mathrm{Out}(\pi(U^-)) \subseteq C \cup B = B \cup C$. To show that $\pi(\mathrm{Out}(U)) \subseteq B \cup C$, note that $\pi(\mathrm{Out}(U)) \subseteq [\pi(\mathrm{Out}(U)) \backslash \pi(\mathrm{Out}(U^-))] \cup \pi(\mathrm{Out}(U^-)) \subseteq \pi(\mathrm{Out}(U) \backslash \mathrm{Out}(U^-)) \cup \pi(\mathrm{Out}(U^-))$. By Lemma 11 we have $\pi(\mathrm{Out}(U) \backslash \mathrm{Out}(U^-)) \subseteq \mathrm{Out}(\pi(U^-))$ and, hence, we get that $\pi(\mathrm{Out}(U)) \subseteq B \subseteq B \cup C$. So we have $A \subseteq B \cup C$.

Let us finally show $B \cup C \subseteq A$. Clearly we have $C \subseteq A$. To show $B \subseteq A$ note first that by Lemma 11 we have $\mathrm{Out}(U^-) \subseteq \mathrm{Out}(U)$ and hence $\pi(\mathrm{Out}(U^-)) \subseteq \pi(\mathrm{Out}(U)) \subseteq A$, so it only remains to show that $\mathrm{Out}(\pi(U^-)) \subseteq A$. Pick $\tilde{u} \in \mathrm{Out}(\pi(U^-))$. We consider two cases:

Case (i): $\tilde{u} \notin C(\pi(\tilde{m}_p))$. By our assumption $\tilde{u} \in \mathrm{Out}(\pi(U^-))$ we have $\tilde{u} \notin \pi(U^-)$. Therefore we have $\tilde{u} \notin \pi(U^-) \cup C(\pi(\tilde{m}_p)) = \pi(U)$. However, since $\tilde{u} \in \mathrm{Out}(\pi(U^-))$ we have for each $l \in \{1, \ldots, n-1\}$ that $\tilde{u} + \tilde{e}_l \in \pi(U^-) \subseteq \pi(U)$ and hence $\tilde{u} \in \mathrm{Out}(\pi(U)) \subseteq A$.

Case (ii): $\tilde{u} \in C(\pi(\tilde{m}_p))$. In this case we shall show that $\tilde{u} \in \pi(\mathrm{Out}(U))$, thus establishing that in both cases, $\tilde{u} \in A$. Let $\tilde{x} = (\tilde{u}, m_{pn} - 1)$. We will show that $\tilde{x} \in \mathrm{Out}(U)$. By definition of $\tilde{x}$ we have $\tilde{x} \notin C(\tilde{m}_p)$. If now $\tilde{x} \in U^-$ then $\tilde{u} = \pi(\tilde{x}) \in \pi(U^-)$, which contradicts our assumption that $\tilde{u} \in \mathrm{Out}(\pi(U^-))$. Hence $\tilde{x} \notin U^- \cup C(\tilde{m}_p) = U$.

For $l \in \{1, \ldots, n\}$ we must show that $\tilde{x} + \tilde{e}_l \in U$: If $l = n$, then since $\tilde{u} \in C(\pi(\tilde{m}_p))$, we have $\tilde{x} + \tilde{e}_l = \tilde{x} + \tilde{e}_n = (\tilde{u}, m_{pn}) \in C(\tilde{m}_p) \subseteq U$.

If $l \in \{1, \ldots, n-1\}$, then we have $\tilde{u} + \tilde{e}_l \in \pi(U^-)$ and hence $\tilde{u} + \tilde{e}_l \in \pi(C(m_i))$ for some $i < p$. Since now $\tilde{x} + \tilde{e}_l = (\tilde{u} + \tilde{e}_l, m_{pn} - 1)$ and $m_{pn} - 1 \geq m_{in}$, we have $\tilde{x} + \tilde{e}_l \in C(\tilde{m}_i) \subseteq U^- \subseteq U$.

Thus, in this case where $\tilde{u} \in C(\pi(\tilde{m}_p))$, $\tilde{x} = (\tilde{u}, m_{pn} - 1) \in \mathrm{Out}(U)$ and hence $\tilde{u} = \pi(\tilde{x}) \in \pi(\mathrm{Out}(U)) \subseteq A$, as required. □

By induction on $p \geq n - 1$, we get from Lemma 12 the following corollary:

Corollary 13. *Let $p \geq n - 1$ and $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^n$ be such that*

$$P_n(\tilde{m}_1) \leq \cdots \leq P_n(\tilde{m}_{n-1}) < \cdots < P_n(\tilde{m}_p). \tag{4}$$

For $i \in \{n-2, \ldots, p\}$ let $U_i = \bigcup_{l=1}^{i} C(\tilde{m}_l)$, and for $i \in \{n-1, \ldots, p\}$ let $A_i = \mathrm{Out}(\pi(U_i)) \backslash \mathrm{Out}(\pi(U_{i-1}))$. If $U = U_p$ is the filter generated by all $\tilde{m}_i$ then $\pi(\mathrm{Out}(U)) \cup \mathrm{Out}(\pi(U)) = \bigcup_{i=n-1}^{p} A_i$ where the union on the right is disjoint.

Let $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^n$. By the definition of Out, we see that each element of $\mathrm{Out}(U) \backslash \mathrm{Out}(U^-)$ must be i-adjacent to $C(\tilde{m}_p)$ for some i, since otherwise, that element would be in $\mathrm{Out}(U^-)$. Hence we have the following lemma.

Lemma 14. *Every point in $\mathrm{Out}(U) \backslash \mathrm{Out}(U^-)$ is adjacent to $C(\tilde{m}_p)$.*

We now have the basic tools to get some more descriptive results:

Lemma 15. $\max_{U\in\mathscr{C}_p^n}(|\mathrm{Out}(U)|+|\mathrm{Out}(\pi(U))|)=c_n(p+1)$.

Proof. Let $\tilde{m}_1,\dots,\tilde{m}_{p+1}\in\mathbb{N}_0^n$. We may assume $m_{p+1n}=\max_i\{m_{i/n}\}$ to hold. By Lemma 11(iii) we have $|\mathrm{Out}(U^+)|\le|\mathrm{Out}(U)|+|\mathrm{Out}(\pi(U))|$ for all $U^+\in\mathscr{C}_{p+1}^n$, and hence $c_n(p+1)\le\max_{U\in\mathscr{C}_p^n}(|\mathrm{Out}(U)|+|\mathrm{Out}(\pi(U))|)$.

To prove the reverse inequality, we shall construct, for any $\tilde{m}_1,\dots,\tilde{m}_p\in\mathbb{N}_0^n$, a point $\tilde{m}_{p+1}$ such that the filter U^+ generated by $\tilde{m}_1,\dots,\tilde{m}_{p+1}$ satisfies $|\mathrm{Out}(U^+)|\ge|\mathrm{Out}(U)|+|\mathrm{Out}(\pi(U))|$. Thus, letting $\tilde{m}_1,\dots,\tilde{m}_p$ range over all such p-tuples, the desired inequality follows.

Let $\tilde{m}_1,\dots,\tilde{m}_p\in\mathbb{N}_0^n$ and let $m=\max\{m_{in}:1\le i\le p\}$. If $\tilde{x}\in\mathrm{Out}(U)$, then by Remark 8 we have $x_n+1=m_{in}\le m$ for some i and hence

$$x_n\le m-1 \quad \text{for all } \tilde{x}\in\mathrm{Out}(U). \tag{5}$$

Let $\tilde{m}_{p+1}=(m+1)\tilde{e}_n^n$ and hence $U^+=\bigcup_{i=1}^{p+1}C(\tilde{m}_i)$. We have by (5) that $\mathrm{Out}(U)\cap C(\tilde{m}_{p+1})=\emptyset$ and hence $\mathrm{Out}(U)\cap U^+=\emptyset$, so by definition of Out we have

$$\mathrm{Out}(U)\subseteq\mathrm{Out}(U^+). \tag{6}$$

Let $f:\mathbb{N}_0^{n-1}\to\mathbb{N}_0^n$ be the injective map defined by $\tilde{u}\mapsto(\tilde{u},m)$. Having now constructed $\tilde{m}_{p+1}$ and f, we assert that $\mathrm{Out}(U^+)$ contains the union of $\mathrm{Out}(U)$ and $f(\mathrm{Out}(\pi(U)))$; that this union is disjoint and that the image of f is disjoint from U^+ is easily seen by (5) and the definition of f. To see that $f(\mathrm{Out}(\pi(U)))$ is contained in $\mathrm{Out}(U^+)$, note that for $\tilde{u}\in\mathrm{Out}(\pi(U))$, $f(\tilde{u})$ is always n-adjacent to $C(\tilde{m}_{p+1})$. To see that for $\tilde{u}\in\mathrm{Out}(\pi(U))$ and $l<n$, $f(\tilde{u})$ is l-adjacent to U^+, choose an i such that $\tilde{u}$ is l-adjacent to $\pi(C(\tilde{m}_i))$. Then $f(\tilde{u})$ is l-adjacent to $C(\tilde{m}_i)$, and the assertion follows together with (6). □

Remark. Let $\tilde{m}_1,\dots,\tilde{m}_p\in\mathbb{N}_0^n$ be such that $|\mathrm{Out}(U)|=c_n(p)$ and $m_{pn}\ge m_{in}$ for all i. For $m=\max_i\{m_{in}\}$ and $\tilde{m}_p'=(m+1)\tilde{e}_n$, let U' be the filter generated by $\tilde{m}_1,\dots,\tilde{m}_{p-1},\tilde{m}_p'$. As we saw in the previous proof, we have $|\mathrm{Out}(U')|\ge|\mathrm{Out}(U^-)|+|\mathrm{Out}\pi(U^-))|\ge|\mathrm{Out}(U)|=c_n(p)$. Hence, when considering a filter U with $|\mathrm{Out}(U)|=c_n(p)$, we can assume $\tilde{m}_p=(m+1)\tilde{e}_n$, and therefore in particular, we can assume $m_{pn}>m_{in}$ for all $i\ne p$, which is what we will do in the proof of our next theorem to come.

By Lemma 15 and (1) we have

$$c_n(p+1)\le c_n(p)+c_{n-1}(p). \tag{7}$$

Note that if, for fixed n, we think of c_n as a sequence, $(c_n(0),c_n(1),\dots)$, then (7) says that the difference sequence of the sequence c_n is bounded by the sequence c_{n-1}. So, for instance, if c_{n-1} can be bounded by a polynomial of some degree d, then c_n can be bounded by a polynomial of degree $d+1$. The next theorem will give a stronger result:

the difference sequence of c_n is bounded by a constant multiple of c_{n-2}. This will then allow us to bound the sequence c_n above by polynomials whose degrees increase by 1 as n is increased by 2 and which, in fact, have the same degrees as do the lower bounds of Lemma 6.

Theorem 16. *For integers $n > 2$ and $p \geq 1$ we have $c_n(p+2) \leq c_n(p+1) + (n-1)c_{n-2}(p)$.*

Proof. Since $c_n(\cdot)$ is finite we may assume we have $\tilde{m}_1, \ldots, \tilde{m}_{p+1} \in \mathbb{N}_0^n$ such that $c_n(p+2) = |\mathrm{Out}(U^+)| + |\mathrm{Out}(\pi(U^+))|$. We may assume $m_{p+1n} > m_{in}$ for all $i \leq p$. By Lemmas 10 and 12 we have

$$\begin{aligned} c_n(p+2) &= |\mathrm{Out}(U^+)| + |\mathrm{Out}(\pi(U^+))| \\ &= |\pi(\mathrm{Out}(U^+)) \cup \mathrm{Out}(\pi(U^+))| \\ &= |\pi(\mathrm{Out}(U)) \cup \mathrm{Out}(\pi(U))| + |\mathrm{Out}(\pi(U^+)) \backslash \mathrm{Out}(\pi(U))| \\ &= |\mathrm{Out}(U)| + |\mathrm{Out}(\pi(U))| + |\mathrm{Out}(\pi(U^+)) \backslash \mathrm{Out}(\pi(U))| \\ &\leq c_n(p+1) + |\mathrm{Out}(\pi(U^+)) \backslash \mathrm{Out}(\pi(U))|. \end{aligned}$$

By Lemma 14, all the points in $\mathrm{Out}(\pi(U^+)) \backslash \mathrm{Out}(\pi(U))$ are adjacent to the orthant $C(\pi(\tilde{m}_{p+1}))$. Hence if we let B_i be the set of points in $\mathrm{Out}(\pi(U^+))$ that are i-adjacent to $C(\pi(\tilde{m}_{p+1}))$, then we have $\mathrm{Out}(\pi(U^+)) \backslash \mathrm{Out}(\pi(U)) \subseteq \bigcup_{i=1}^{n-1} B_i$ and therefore $|\mathrm{Out}(\pi(U^+)) \backslash \mathrm{Out}(\pi(U))| \leq \sum_{i=1}^{n-1} |B_i|$. We will show that $|B_i| \leq c_{n-2}(p)$ for all $i \in \{1, \ldots, n\}$.

Since $\tilde{u} \in B_i$ is i-adjacent to $C(\pi(\tilde{m}_{p+1}))$, we have by Remark 8 that $u_i = m_{p+1i} - 1$. For $i \in \{1, \ldots, n-1\}$ and $a \in \mathbb{N}_0$ let $\boldsymbol{P}_i(a) = \mathbb{N}_0^{i-1} \times \{a\} \times \mathbb{N}_0^{n-i-1} \subseteq \mathbb{N}_0^{n-1}$. By Remark 8 we have $B_i \subseteq \boldsymbol{P}_i(m_{p+1i} - 1)$, which we shall simply write as $\boldsymbol{P}_i$.

Let $\tilde{u} \in B_i$. Since $\tilde{u} \in \mathrm{Out}(\pi(U^+))$, we have in particular $\tilde{u} \notin \pi(U)$. Let $j \in \{1, \ldots, \hat{i}, \ldots, n-1\}$. We clearly have $C(\pi(\tilde{m}_{p+1})) \cap \boldsymbol{P}_i = \emptyset$, so since $\tilde{u} + \tilde{e}_j \in \boldsymbol{P}_i$, we have $\tilde{u} + \tilde{e}_j \notin C(\pi(\tilde{m}_{p+1}))$. Since, on the other hand, $\tilde{u} \in \mathrm{Out}(\pi(U^+))$ and hence $\tilde{u} + \tilde{e}_j \in \pi(U^+)$, we have that $\tilde{u} + \tilde{e}_j \in \pi(U^+) \backslash C(\pi(\tilde{m}_{p+1})) \subseteq \pi(U)$. We therefore have for all $\tilde{u} \in B_i$ that

$$\begin{aligned} &\tilde{u} \notin \pi(U) \cap \boldsymbol{P}_i, \\ &\tilde{u} + \tilde{e}_j \in \pi(U) \cap \boldsymbol{P}_i \quad \text{for all } j \in \{1, \ldots, \hat{i}, \ldots, n-1\}. \end{aligned} \tag{8}$$

Note that for any $a \in \mathbb{N}_0$ the map π_i restricted to $\boldsymbol{P}_i(a) \subseteq \mathbb{N}_0^{n-1}$ is injective. We get therefore from (8) that the following holds for all $\tilde{u} \in B_i$.

$$\begin{aligned} &\pi_i(\tilde{u}) \notin \pi_i(\pi(U) \cap \boldsymbol{P}_i), \\ &\pi_i(\tilde{u}) + \tilde{e}_l^{n-2} \in \pi_i(\pi(U) \cap \boldsymbol{P}_i) \quad \text{for all } l \in \{1, \ldots, n-2\}. \end{aligned} \tag{9}$$

One can easily get that for any $a \in \mathbb{N}_0$ and any $\tilde{b} \in \mathbb{N}_0^{n-1}$ the following holds in $\mathbb{N}_0^{n-2}$:

$$\pi_i(C(\tilde{b}) \cap \boldsymbol{P}_i(a)) = \begin{cases} \emptyset & \text{if } a < b_i, \\ C(\pi_i(\tilde{b})) & \text{if } a \leq b_i. \end{cases}$$

Therefore, since $U = \bigcup_{i=1}^{p} C(\tilde{m}_i) \subseteq \mathbb{N}_0^n$, we see that $H_i = \pi_i(\pi(U) \cap \boldsymbol{P}_i)$ is a union of at most p orthants in $\mathbb{N}_0^{n-2}$. Since $B_i \subseteq \boldsymbol{P}_i$ and π_i restricted to $\boldsymbol{P}_i$ is injective, we get by (9) that

$$\begin{aligned}|B_i| &\leq |\{\tilde{v} \in \mathbb{N}_0^{n-2} : \tilde{v} \notin H_i \text{ and } \tilde{v} + \tilde{e}_l^{n-2} \in H_i \text{ for all } 1 \leq l \leq n-2\}| \\ &\leq c_{n-2}(p).\end{aligned}$$

So we finally have $|\mathrm{Out}(\pi(U^+)) \backslash \mathrm{Out}(\pi(U))| \leq (n-1)c_{n-2}(p)$. □

Lemma 17. *For $n < 4$ and $p \geq n$ we have*

$$c_n(p) = (n-1)(p-n) + 1. \tag{10}$$

Proof. For $n = 1$ we get by definition $c_1(p) = 1$ for all $p \geq 1$.

For both $n = 2$ and $n = 3$, we have $c_n(n) = 1$ by Lemma 9, as required. Moreover for $n = 2$, (7) tells us that the difference between successive values $c_n(p)$ and $c_n(p+1)$ is at most $c_{n-1}(p) = c_1(p) = 1 = n-1$, while for $n = 3$ Theorem 16 tells us that these differences are at most $(n-1)c_{n-2}(p) = 2c_1(p) = 2 = n-1$. So in each case, the function c_n has the value given by (10) for $p = n$ and does not increase faster than the left-hand side of (10). But Lemma 4 shows that in each of these cases, c_n has at least the value given by (10); hence the lemma follows. □

Corollary 18. *For $n \geq 2$ and $p \geq 1$ we have*

$$c_n(p) < (2p)^{[n/2]}. \tag{11}$$

Proof. By Lemma 9 we have (11) for all $n \geq 2$ and $p \in \{1, 2\}$. By Lemma 17 we have $c_2(p) = p - 1 < (2p)^{[2/2]}$ and $c_3(p) = 2p - 5 < (2p)^{[3/2]}$. So we have (11) for $n \in \{2, 3\}$ and $p \geq 1$. We have in particular (11) if $n + p \leq 6$. We will proceed by induction on $n + p$. Let $N > 6$ be given and assume we have (11) if $n+p < N$. Let $n \geq 3$ and $p \geq 3$ be such that $n + p = N$. Note in the calculation below that for any positive integer m we have $2[m/2] \geq m - 1$ and $[(m-2)/2] = [m/2] - 1$. Also we use the "binomial inequality" $(x+2)^m > x^m + 2mx^{m-1}$ for all real $x \geq 0$. We have now by Theorem 16

$$\begin{aligned}c_n(p) &\leq c_n(p-1) + (n-1)c_{n-2}(p-2) \\ &< (2p-2)^{[n/2]} + (n-1)(2p-4)^{[(n-2)/2]} \\ &\leq (2p-2)^{[n/2]} + 2\left[\frac{n}{2}\right](2p-2)^{[n/2]-1} \\ &< (2p)^{[n/2]}.\end{aligned}$$

Hence we have (11) for all $n \geq 2$ and $p \geq 1$. □

By Lemma 6 one can easily show that

$$\frac{c_n(p)}{p^{[n/2]}} \geq ((n+1)[n/2])^{-[n/2]},$$

for all $p \geq n \geq 2$. With this in mind together with Corollary 18 we have:

Theorem 19. *Consider $c_n(p)$ for positive integers $p \geq n$. There exist positive functions ε, $K : \mathbb{N} \to \mathbb{R}^+$ of one variable n such that*

$$\varepsilon(n)p^{[n/2]} < c_n(p) < K(n)p^{[n/2]}$$

for all $p \geq n$, which more compactly is denoted by $c_n(p) = \Theta(p^{[n/2]})$.

3. Technical results on outer corner points

The main result we state in this section is the following lemma, which will be used in the last section in which we consider the case $n = 4$.

Proposition 20. *There are $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^n$ satisfying the following:*
- $|\mathrm{Out}(U)| = c_n(p)$.
- $\{\tilde{m}_1, \ldots, \tilde{m}_p\}$ *forms an antichain in* $\mathbb{N}_0^n$.
- *Writing $\tilde{m}_1, \ldots, \tilde{m}_p$ as column vectors in the following $n \times p$ matrix M, we have $M = [\tilde{m}_1 | \cdots | \tilde{m}_p] = [D|(p-n+1)I]$ where D is an $n \times (p-n)$ matrix, each row of which is a permutation of the positive integers $1, 2, \ldots, p-n$ and where I is the $n \times n$ identity matrix.*

Remark. We see in particular that for given n and $p \geq n$, there are $((p-n)!)^n$ possibilities for the matrix D. So in order to calculate $c_n(p)$, we "only" have to find the maximal $|\mathrm{Out}(U)|$ for these $((p-n)!)^n$ possible matrices M in the above lemma.

Many of the following lemmas are easy, and we just sketch the proofs.

Lemma 21. *If $\tilde{m}_1, \ldots, \tilde{m}_p, \tilde{m}'_1, \ldots, \tilde{m}'_p \in \mathbb{N}_0^n$ and for all $l \in \{1, \ldots, n\}$ and $a, b \in \{1, \ldots, p\}$ we have*

$$P_l(\tilde{m}_a) \geq P_l(\tilde{m}_b) \Leftrightarrow P_l(\tilde{m}'_a) \geq P_l(\tilde{m}'_b), \tag{12}$$

then we have that $|\mathrm{Out}(U)| = |\mathrm{Out}(U')|$.

Proof (*sketch*). Since for real numbers x and y we have that $x = y \Leftrightarrow x \geq y$ and $x \leq y$, we get from (12) that equality holds on the right side if and only if it holds on the left. For $\tilde{x} \in \mathrm{Out}(U)$ there is for each $l \in \{1, \ldots, n\}$ an $a_l \in \{1, \ldots, p\}$ such that $\tilde{x}$ is l-adjacent to $C(\tilde{m}_{a_l})$. By Remark 8 $\tilde{x} = (P_1(\tilde{m}_{a_1}) - 1, \ldots, P_n(\tilde{m}_{a_n}) - 1)$. If we now let $\tilde{x}' = (P_1(\tilde{m}'_{a_1}) - 1, \ldots, P_n(\tilde{m}'_{a_n}) - 1)$, then we derive two things from (12). On the one hand, $\tilde{x} \mapsto \tilde{x}'$ is a well-defined bijection, and, on the other hand, $\tilde{x} \in \mathrm{Out}(U) \Leftrightarrow \tilde{x}' \in \mathrm{Out}(U')$. Hence we have the lemma. $\square$

If $k \in \mathbb{N}$ and $\tilde{x} = (x_1, \dots, x_n)$, then $k\tilde{x}$ will always mean $(kx_1, \dots, kx_n)$. For $k \in \mathbb{N}$ and $\tilde{m}_1, \dots, \tilde{m}_p \in \mathbb{N}_0^n$ let $U^k = \bigcup_{i=1}^{p} C(k\tilde{m}_i)$.

Letting $\tilde{m}'_i = k\tilde{m}_i$ in Lemma 21, we see that $\tilde{m}_1, \dots, \tilde{m}_p, \tilde{m}'_1, \dots, \tilde{m}'_p \in \mathbb{N}_0^n$ satisfy (12), and hence $|\mathrm{Out}(U)| = |\mathrm{Out}(U^k)|$. The map $\tilde{x} \mapsto \tilde{x}'$ defined in the proof of Lemma 21 can in this case be given by the formula

$$s_k(\tilde{x}) = k\tilde{x} + (k-1)(1, 1, \dots, 1). \tag{13}$$

Let $B(\tilde{a}, \tilde{b}) = \{\tilde{y} \in \mathbb{N}_0^n : \tilde{a} \le \tilde{y} \le \tilde{b}\}$. For $\tilde{x} \in \mathrm{Out}(U)$ and $\tilde{y} \in B(k\tilde{x}, s_k(\tilde{x}))$, we have in particular $\tilde{y} \le s_k(\tilde{x}) \in \mathrm{Out}(U^k)$. But since $\mathrm{Out}(U^k)$ is the set of maximal elements of $\mathbb{N}_0^n \backslash U^k$, which is a co-filter, we have $\tilde{y} \in \mathbb{N}_0^n \backslash U^k$; that is $\tilde{y} \notin U^k$.

Again, if $\tilde{y} \in B(k\tilde{x}, s_k(\tilde{x}))$, we have $\tilde{y} \ge k\tilde{x}$ and therefore $\tilde{y} + k\tilde{e}_l \ge k\tilde{x} + k\tilde{e}_l = k(\tilde{x} + \tilde{e}_l) \in kU \subseteq U^k$. Hence, $\tilde{y} + k\tilde{e}_l \in U^k$ for all $l \in \{1, \dots, n\}$.

We summarize in

Lemma 22. *For $k \in \mathbb{N}$ and $\tilde{m}_1, \dots, \tilde{m}_p \in \mathbb{N}_0^n$ we have:*

(i) *The map $s_k : \mathrm{Out}(U) \to \mathrm{Out}(U^k)$ defined in (13) is a bijection.*

(ii) *If $\tilde{x} \in \mathrm{Out}(U)$, then for every $\tilde{y} \in B(k\tilde{x}, s_k(\tilde{x}))$, we have*
- $\tilde{y} \notin U^k$,
- $\tilde{y} + k\tilde{e}_l \in U^k$ *for all* $l \in \{1, \dots, n\}$.

Lemma 23. *For $n \ge 1$ and $p \ge n$ we have $c_n(p+1) \ge c_n(p) + n - 1$.*

Proof. Let $p \in \mathbb{N}$ be such that $p \ge n$. Let $\tilde{m}_1, \dots, \tilde{m}_p \in \mathbb{N}_0^n$ be such that $|\mathrm{Out}(U)| = c_n(p)$. By Lemma 22 we also have $|\mathrm{Out}(U^2)| = c_n(p)$. Since $p \ge n$ $\mathrm{Out}(U^2)$ contains at least one point which by Lemma 22 has the form $s_2(\tilde{m})$ for some $\tilde{m} \in \mathrm{Out}(U)$. If $U' = U^2 \cup \{s_2(\tilde{m})\}$, then U' is a filter generated by $2\tilde{m}_1, \dots, 2\tilde{m}_p, s_2(\tilde{m})$. We want to show that $|\mathrm{Out}(U')| \ge c_n(p) + n - 1$.

If $\tilde{x} \in \mathrm{Out}(U^2) \backslash \{s_2(\tilde{m})\}$, then $\tilde{x} \notin U^2$ and hence $\tilde{x} \notin U'$. From this and the fact that $U^2 \subseteq U'$, we get by definition of Out that

$$\mathrm{Out}(U^2) \backslash \{s_2(\tilde{m})\} \subseteq \mathrm{Out}(U'). \tag{14}$$

For each $i \in \{1, \dots, n\}$ let $\tilde{x}_i = 2\tilde{m} + \tilde{e}_1 + \cdots + \widehat{\tilde{e}_i} + \cdots + \tilde{e}_n$. Note that the only even coordinate of $\tilde{x}_i$ is $P_i(\tilde{x}_i)$; hence the n points $\tilde{x}_i$ are all distinct. Let S be the set of these n points. We will show that $S \subseteq \mathrm{Out}(U')$: Since $S \subseteq B(2\tilde{m}, s_2(\tilde{m}))$, we have by Lemma 22 that $S \cap U^2 = \emptyset$, and, since $s_2(\tilde{m}) \notin S$, we have $S \cap U' = \emptyset$. Again, by Lemma 22 and the definition of S, we also have that $\tilde{x}_i + \tilde{e}_l \in U'$ for all $i, l \in \{1, \dots, n\}$, and hence

$$S \subseteq \mathrm{Out}(U'). \tag{15}$$

Finally, note that for all $i \in \{1, \dots, n\}$, we have that $\tilde{x}_i + \tilde{e}_i = s_2(\tilde{m}) \notin U^2$ and hence

$$S \cap \mathrm{Out}(U^2) = \emptyset. \tag{16}$$

From (14)–(16) we get

$$\begin{aligned} c_n(p+1) &\geq |\mathrm{Out}(U')| \\ &\geq |(\mathrm{Out}(U^2)\backslash\{s_2(\tilde{m})\})\cup S| \\ &\geq |\mathrm{Out}(U^2)\backslash\{s_2(\tilde{m})\}| + |S| \\ &= (c_n(p)-1)+n. \qquad \square \end{aligned}$$

Remark. Basically, what we did in the above proof was to change one outer corner point of U^2 into n distinct outer corner points by adding an orthant to U at that particular outer corner point. This method could be used to prove a more general result: For $p \geq n$ let U_p be such that $|\mathrm{Out}(U_p)| = c_n(p)$. For $q \geq n$ and sufficiently large k, one can again change one outer corner point $\tilde{x}$ of U_p^k to $c_n(q)$ distinct ones by adding $q-n$ orthants appropriately in the cavity created by the n orthants that $\tilde{x}$ is adjacent to. In this way one can get $c_n(p+q-n) \geq c_n(p) + c_n(q) - 1$ for all $p, q \geq n$.

Proof Proposition 20. Let $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^n$ be such that $|\mathrm{Out}(U)| = c_n(p)$. Assume for a moment that the $\tilde{m}$ are labelled in such a way that $P_1(\tilde{m}_1) \leq \cdots \leq P_1(\tilde{m}_p)$. Let $\tilde{m}'_1, \ldots, \tilde{m}'_p \in \mathbb{N}_0^n$ be defined by $\tilde{m}'_i = (i, \pi(\tilde{m}_i))$ for all $i \in \{1, \ldots, p\}$. If now $\tilde{x} \in \mathrm{Out}(U)$, let $i_{\tilde{x}} \in \{1, \ldots, p\}$ be the least positive integer such that $\tilde{x}$ is 1-adjacent to $C(\tilde{m}_{i_{\tilde{x}}})$. In this case, it is not too hard to show that $\tilde{x}' = (i_{\tilde{x}} - 1, \pi(\tilde{x})) \in \mathbb{N}_0^n$ is in fact a member of $\mathrm{Out}(U')$. By Lemma 10, $\tilde{x} \mapsto \tilde{x}'$ is injective, and hence $|\mathrm{Out}(U)| \leq |\mathrm{Out}(U')|$. We have shown that if $|\mathrm{Out}(U)| = c_n(p)$, then we can assume the first coordinate of $\tilde{m}_1, \ldots, \tilde{m}_p$ to be precisely the integers $1, 2, \ldots, p$. In the same way we can show the same for every other coordinate of $\tilde{m}_1, \ldots, \tilde{m}_p$. Hence there are $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^n$ such that $|\mathrm{Out}(U)| = c_n(p)$ and $\{P_l(\tilde{m}_i) : i = 1, \ldots, p\} = \{1, \ldots, p\}$ for all $l \in \{1, \ldots, n\}$ and, hence, $\tilde{m}_1, \ldots, \tilde{m}_p \in \{1, \ldots, p\}^n$.

If now $i \in \{1, \ldots, p\}$, there must be an $\tilde{x} \in \mathrm{Out}(U)$ adjacent to $C(\tilde{m}_i)$, since otherwise we would have $\mathrm{Out}(U) \subseteq \mathrm{Out}(\langle \tilde{m}_1, \ldots, \widehat{\tilde{m}_i}, \ldots, \tilde{m}_p \rangle)$ and hence $c_n(p) = |\mathrm{Out}(U)| \leq c_n(p-1)$, which would contradict Lemma 23. This fact that every $C(\tilde{m}_i)$ has at least one outer corner point adjacent to it implies that at most one coordinate of each $\tilde{m}_i$ can be the maximal number p. We can therefore further assume $\tilde{m}_1, \ldots, \tilde{m}_p \in \{1, \ldots, p\}^n$ to be labelled in such a way that $P_l(\tilde{m}_{p+l-n}) = p$ for all $l \in \{1, \ldots, n\}$.

Now, consider the filter U^* of $\mathbb{N}_0^n$ generated by $\tilde{m}_1, \ldots, \tilde{m}_{p-n}, p\tilde{e}_1, \ldots, p\tilde{e}_n$. Clearly $U \subseteq U^*$, and moreover, since each $\tilde{x} \in \mathrm{Out}(U)$ is l-adjacent to some $C(\tilde{m}_i)$ for all l, we get by Remark 8 that $\tilde{x} \notin p\tilde{e}_j$ for all $j \in \{1, \ldots, n\}$. Hence $\mathrm{Out}(U) \subseteq \mathrm{Out}(U^*)$. We can therefore further assume $\tilde{m}_{p+l-n} = (p\tilde{e}_l)$ for all $l \in \{1, \ldots, n\}$.

We now have for $l \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p-n\}$ that $P_l(\tilde{m}_j) \in \{1, \ldots, p-1\}$ and $|\{P_l(\tilde{m}_j) : j = 1, \ldots, p-n\}| = p-n$. Therefore we can by Lemma 21 assume $\{P_l(\tilde{m}_j) : j = 1, 2, \ldots, p-n\} = \{1, 2, \ldots, p-n\}$ for all $l \in \{1, \ldots, n\}$. Hence we have Proposition 20. $\square$

4. The case $n = 4$

From Theorem 16 and Lemma 17 we already have

$$c_4(p+1) - c_4(p) \leq 3 \cdot c_2(p-1) = 3(p-2)$$

for all $p \geq 4$. We will show that there is a smaller bound than this. First we must state the following lemma.

Lemma 24. *For a filter U in $\mathbb{N}_0^3$ generated by p elements, there are at most $p-1$ points of* Out(U) *that are adjacent to a given octant $C(\tilde{m})$, where $\tilde{m}$ is one of the generators for U.*

Idea of proof. This statement is in fact easy to believe if we consider the example given in Fig. 2. If we imagine looking along the z-axis in $\mathbb{N}_0^3$ (the circled X-sign by the letter z in the figure means that the z-axis is perpendicular to and is pointing into the page), then we see 21 outer corner points adjacent to a given $C(\tilde{m})$, where $\tilde{m}$ is one of the 22 generators. The idea of this lemma is that the octants that share outer corner points with $C(\tilde{m})$ lie in a "cycle" around $C(\tilde{m})$, with one outer corner point falling between each two successive members of this cycle. Hence there are the same number of outer corner points as elements in the cycle. □

From this lemma we deduce the following corollary:

Corollary 25. *For $p \geq 4$ we have $c_4(p+1) \leq c_4(p) + p - 1$.*

Proof. By Lemma 15, there is a filter U of $\mathbb{N}_0^4$ generated by $\tilde{m}_1, \ldots, \tilde{m}_p \in \mathbb{N}_0^4$ such that $|\text{Out}(U)| + |\text{Out}(\pi(U))| = c_4(p+1)$. By Lemmas 10 and 12, we get for this filter U

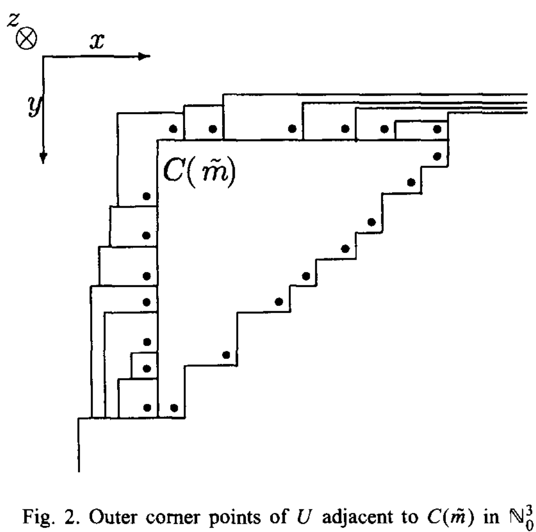

Fig. 2. Outer corner points of U adjacent to $C(\tilde{m})$ in $\mathbb{N}_0^3$.

that

$$\begin{aligned}
c_4(p+1) &= |\mathrm{Out}(U)| + |\mathrm{Out}(\pi(U))| \\
&= |\pi(\mathrm{Out}(U)) \cup \mathrm{Out}(\pi(U))| \\
&= |\pi(\mathrm{Out}(U^-)) \cup \mathrm{Out}(\pi(U^-))| + |\mathrm{Out}(\pi(U))\backslash\mathrm{Out}(\pi(U^-))| \\
&\le c_4(p) + |\mathrm{Out}(\pi(U))\backslash\mathrm{Out}(\pi(U^-))|.
\end{aligned}$$

By Lemma 14, all points of $\mathrm{Out}(\pi(U))\backslash\mathrm{Out}(\pi(U^-))$ are adjacent to $C(\pi(\tilde{m}_p))$. By Lemma 24, there can be at most $p-1$ such points, so the last expression in the above display is $\le c_4(p)+p-1$. □

Note that a function with value 1 at $p=4$ and whose difference function is $p-1$ for $p\ge 4$, is $(p^2-3p-2)/2$. By this observation, the preceding corollary for the bound on the difference, and Lemma 9, we have the following:

Corollary 26. *For all $p\ge 4$ we have*

$$c_4(p) \le \frac{(p^2-3p-2)}{2}.$$

Moreover, if the equality holds for some p, then equality holds for all smaller p.

Consider the filter U of $\mathbb{N}_0^4$ generated by the 12 points:

$$(9,0,0,0),(0,9,0,0),(0,0,9,0),(0,0,0,9),(6,7,4,1),(2,3,8,5),$$
$$(5,8,3,2),(1,4,7,6),(8,5,2,3),(4,1,6,7),(7,6,1,4),(3,2,5,8).$$

If I is the monomial ideal of $R=k[X_1,X_2,X_3,X_4]$ corresponding to the filter U, and m is the maximal ideal of R generated by the indeterminates, then one can calculate $\dim_k((m:I)/I)=|\mathrm{Out}(U)|$ using a computer algebra system (i.e. [2]), and find $|\mathrm{Out}(U)|=53$ for this filter U. By Corollary 26 we therefore have

Lemma 27. $c_4(p)=(p^2-3p-2)/2$ *for* $p\in\{4,5,\dots,12\}$

In the same way, we get for the filter U of $\mathbb{N}_0^4$ generated by the 13 points:

$$(10,0,0,0),(0,10,0,0),(0,0,10,0),(0,0,0,10),$$
$$(7,8,4,1),(2,4,9,5),(6,9,3,2),(1,5,8,6),$$
$$(9,6,2,3),(5,1,7,7),(8,7,1,4),(4,3,6,8),(3,2,5,9),$$

that $|\mathrm{Out}(U)|=63$, and hence, again by Corollary 26, we have

Lemma 28. $c_4(13)$ *is equal to either* 63 *or* 64.

In order to determine $c_4(13)$ completely, we need a couple of lemmas.

Lemma 29. *Let U be a filter of $\mathbb{N}_0^4$ generated by $\tilde{m}_1,\dots,\tilde{m}_p \in \mathbb{N}_0^4$ that satisfy the last itemized condition of Proposition 20. If $|\mathrm{Out}(U)| = (p^2-3p-2)/2$ then the following holds*:

(i) *For any $i \in \{1,2,3,4\}$, $\{\pi_i(\tilde{m}_l) : l \in \{1,\dots,p\},\ l \neq p+i-4\}$ forms an antichain in $\mathbb{N}_0^3$.*

(ii) *For any pair $\{i,j\}$ of distinct elements of $\{1,2,3,4\}$, $\{\pi_{ij}(\tilde{m}_l) : l = 1,2,\dots p-4\}$ contains no 3 element chain in $\mathbb{N}_0^2$.*

Proof. By symmetry it suffices to prove the first statement for $i=1$ and the second statement for $\{i,j\}=\{3,4\}$. In that case, in order to simplify index labelling, we will relabel $\tilde{m}_1,\dots,\tilde{m}_p$ as follows:

$$\begin{aligned}
&\tilde{m}_1 = (p-3,0,0,0),\\
&\tilde{m}_2 = (0,p-3,0,0),\\
&\tilde{m}_3 = (0,0,p-3,0),\\
&\{P_i(\tilde{m}_l) : l=4,\dots,p-1\} = \{1,\dots,p-4\} \text{ for all } i \in \{1,2,3,4\},\\
&\tilde{m}_p = (0,0,0,p-3),
\end{aligned} \tag{17}$$

such that (4) is satisfied. Up to the order of the $\tilde{m}_i$, these assumptions imply the last itemized condition of Lemma 20 in the case $n=4$.

For $i \in \{2,\dots,p\}$ let $U_i = \bigcup_{l=1}^{i} C(\tilde{m}_l)$. By our assumption in (17) and (4), Corollary 13 now applies. By Lemmas 14 and 24, we have $|\mathrm{Out}(\pi(U_i)) \backslash \mathrm{Out}(\pi(U_{i-1}))| \le i-1$ for $i>1$, and hence by Lemmas 13 and 10 in this sequence, we get

$$\begin{aligned}
(p^2-3p-2)/2 &= 1 + \sum_{i=4}^{p-1} (i-1)\\
&\ge 1 + \sum_{i=4}^{p-1} |\mathrm{Out}(\pi(U_i)) \backslash \mathrm{Out}(\pi(U_{i-1}))|\\
&= \left| \bigcup_{i=3}^{p-1} \mathrm{Out}(\pi(U_i)) \backslash \mathrm{Out}(\pi(U_{i-1})) \right|\\
&= |\pi(\mathrm{Out}(U_{p-1})) \cup \mathrm{Out}(\pi(U_{p-1}))|\\
&= |\mathrm{Out}(U_{p-1})| + |\mathrm{Out}(\pi(U_{p-1}))|.
\end{aligned} \tag{18}$$

Now, by Lemma 11(iii), we have for $U = U_p$:

$$\begin{aligned}
|\mathrm{Out}(U_{p-1})| + |\mathrm{Out}(\pi(U_{p-1}))| &\ge |\mathrm{Out}(U_p)|\\
&= |\mathrm{Out}(U)|.
\end{aligned} \tag{19}$$

Now if $|\mathrm{Out}(U)| = (p^2 - 3p - 2)/2$, then we see that inequalities in (18) and (19) must be equalities. In particular, this condition on the inequality in (18) means that $|\mathrm{Out}(\pi(U_i))\backslash\mathrm{Out}(\pi(U_{i-1}))| = i-1$ for each $i \in \{4, \ldots, p-1\}$.

So in order to prove both the first and the second statement of Lemma 29, it suffices therefore to show that if either of the statements is false, then there exists an $i \in \{4, \ldots, p-1\}$ such that

$$|\mathrm{Out}(\pi(U_i))\backslash\mathrm{Out}(\pi(U_{i-1}))| \leq i-2. \tag{20}$$

Suppose the first statement is false; i.e. that $\{\pi(\tilde{m}_i) : i = 1, \ldots p-1\}$ does not form an antichain. Therefore there are $i, j \in \{1, \ldots, p-1\}$ with $i \neq j$ and $\pi(\tilde{m}_i) \leq \pi(\tilde{m}_j)$ w.r.t. the natural partial order of $\mathbb{N}_0^3$. By our assumption on $\tilde{m}_1, \ldots, \tilde{m}_p$, we clearly have $i, j \in \{4, \ldots, p-1\}$. We consider the following two cases:

Case 1: $i < j$. Here we have $C(\pi(\tilde{m}_j)) \subseteq C(\pi(\tilde{m}_i)) \subseteq \pi(U_{j-1})$ and hence $\pi(U_{j-1}) \subseteq \pi(U_{j-1})$. We have therefore $\pi(U_j) = \pi(U_{j-1})$, which implies that $\mathrm{Out}(\pi(U_j))\backslash \mathrm{Out}(\pi(U_{j-1})) = \emptyset$, proving (20).

Case 2: $i > j$. Here we have $C(\pi(\tilde{m}_j)) \subseteq C(\pi(\tilde{m}_i))$ and hence

$$\pi(U_i) = \bigcup_{l=1}^{i} C(\pi(\tilde{m}_l)) = \bigcup_{l=1,\, l\neq j}^{i} C(\pi(\tilde{m}_l)).$$

We see that in this case $\pi(U_i)$ is a filter of $\mathbb{N}_0^3$ generated by $i-1$ elements. By Lemma 14 every point of $\mathrm{Out}(\pi(U_i))\backslash\mathrm{Out}(\pi(U_{i-1}))$ is contained in $\mathrm{Out}(\pi(U_i))$ and is adjacent to $C(\pi(\tilde{m}_i))$, which by Lemma 24 is at most $i-2$, again proving (20).

Before we go on to prove the second statement of Lemma 29, we will set forth two helpful lemmas that will only be used within this proof of Lemma 29.

Lemma 30. *Suppose $\tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{N}_0^n$ are such that for every $i \in \{1, \ldots, n\}$*

$$b_i \text{ lies strictly between } a_i \text{ and } c_i. \tag{21}$$

Then every point $\tilde{x}$ adjacent to both $C(\tilde{a})$ and $C(\tilde{c})$ is contained in $C(\tilde{b})$. Consequently, if such $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ are among the generators of a filter U of $\mathbb{N}_0^n$, no point of $\mathrm{Out}(U)$ *is adjacent to both $C(\tilde{a})$ and $C(\tilde{c})$.*

Proof. Assume $\tilde{x}$ is i-adjacent to $\tilde{a}$ and j-adjacent to $\tilde{c}$. Then every coordinate of $\tilde{x}$ other than the ith and jth coordinates is $\geq$the corresponding coordinates of both $\tilde{a}$ and $\tilde{c}$ and, hence, by (21) is also $\geq$the corresponding coordinate of $\tilde{b}$. Moreover, by (21), no coordinate of $\tilde{a}$ can equal the corresponding coordinate of $\tilde{c}$. Hence, by the definition of i-adjacent and Remark 8, i and j must be distinct. By the fact that it is i-adjacent to $\tilde{a}$, $\tilde{x}$ must have j-coordinate $\geq$that of $\tilde{a}$. Thus we have $a_j \leq x_j = c_j - 1$. Since b_j lies strictly between a_j and c_j, it must be $\leq c_j - 1 = x_j$. By the same argument with the roles of i and j interchanged, we likewise have $b_i \leq x_i$, completing the proof that all coordinates of $\tilde{x}$ majorize the corresponding coordinate of $\tilde{b}$; i.e. that $\tilde{x} \in C(\tilde{b})$.

The final assertion of the lemma holds because a member of $\mathrm{Out}(U)$ cannot belong to U, and hence in particular, cannot belong to $C(\tilde{b})$. □

Lemma 31. *Let $\tilde{m}_1,\dots,\dots m_p \in \mathbb{N}_0^n$ and let $U_p = U$ and $U_{p-1} = U^-$. Assume there is a $j \in \{1,\dots,p-1\}$ such that there is no point in $\mathrm{Out}(U_p)$ adjacent to both $C(\tilde{m}_p)$ and $C(\tilde{m}_j)$. If now*

$$U_p^o = \bigcup_{i=1,\, i\neq j}^{p} C(\tilde{m}_i) \quad \text{and} \quad U_{p-1}^o = \bigcup_{i=1,\, i\neq j}^{p-1} C(\tilde{m}_i)$$

then we have $\mathrm{Out}(U_p)\backslash\mathrm{Out}(U_{p-1}) \subseteq \mathrm{Out}(U_p^o)\backslash\mathrm{Out}(U_{p-1}^o)$, and therefore $|\mathrm{Out}(U_p)\backslash \mathrm{Out}(U_{p-1})| \leq p-2$.

Proof. Let $\tilde{x} \in \mathrm{Out}(U_p)\backslash\mathrm{Out}(U_{p-1})$. By Lemma 14, $\tilde{x}$ is adjacent to $C(\tilde{m}_p)$ and hence not adjacent to $C(\tilde{m}_j)$. Therefore $\tilde{x} \in \mathrm{Out}(U_p^o)$.

Note that, in general, if $U \subseteq U' \subseteq U''$, then from the definition of Out, we have $\mathrm{Out}(U) \cap \mathrm{Out}(U'') \subseteq \mathrm{Out}(U')$; so, if $\tilde{x} \in \mathrm{Out}(U_{p-1}^o)$, then $\tilde{x} \in \mathrm{Out}(U_p) \cap \mathrm{Out}(U_{p-1}^o) \subseteq \mathrm{Out}(U_{p-1})$, a contradiction. Hence we have the lemma. □

Let us now go back to proving the second statement of Lemma 29. Let U as before be the filter of $\mathbb{N}_0^4$ generated by $\tilde{m}_1,\dots,\tilde{m}_p$ satisfying the conditions in (17). If $|\mathrm{Out}(U)| = (p^2-3p-2)/2$ then we will show that $M = \{\pi_{34}(\tilde{m}_i)\colon\ i = 4,5,\dots, p-1\} \subseteq \mathbb{N}_0^2$ contains no 3 element chain in $\mathbb{N}_0^2$. To do this assume that we do have a 3 element chain

$$\pi_{34}(\tilde{m}_i) < \pi_{34}(\tilde{m}_j) < \pi_{34}(\tilde{m}_k) \tag{22}$$

in M. By the first statement of Lemma 29, which we now have proved, we have that $\{\pi(\tilde{m}_i), \pi(\tilde{m}_j), \pi(\tilde{m}_k)\}$ forms an antichain in $\mathbb{N}_0^3$ and hence we have

$$P_3(\tilde{m}_i) > P_3(\tilde{m}_j) > P_3(\tilde{m}_k). \tag{23}$$

We want to show that in this situation there is an $l \in \{i,j,k\}$ such that we have $|\mathrm{Out}(\pi(U_l))\backslash\mathrm{Out}(\pi(U_{l-1}))| \leq l-2$. We have three cases to consider. Note that the numerical order of the indices i,j,k is significant because of the assumption (4).

Case 1: $j = \max\{i,j,k\}$. Recall that we have $\tilde{m}_3 = (0,0,p-3,0)$ by the assumption in (17). By (22) and (23) we can apply Lemma 30 to the triple $\{\tilde{a},\tilde{b},\tilde{c}\} = \{\pi(\tilde{m}_3), \pi(\tilde{m}_i), \pi(\tilde{m}_j)\}$. Hence we have that there is no point in $\mathrm{Out}(\pi(U_j))$ adjacent to both $C(\pi(\tilde{m}_j))$ and $C(\pi(\tilde{m}_3))$. If

$$U_j^o = \bigcup_{l=1,\, l\neq 3}^{j} C(\tilde{m}_l) \quad \text{and} \quad U_{j-1}^o = \bigcup_{l=1,\, l\neq 3}^{j-1} C(\tilde{m}_l)$$

and hence

$$\pi(U_j^o) = \bigcup_{l=1,\, l\neq 3}^{j} C(\pi(\tilde{m}_l)) \quad \text{and} \quad \pi(U_{j-1}^o) = \bigcup_{l=1,\, l\neq 3}^{j-1} C(\pi(\tilde{m}_l)),$$

then we have by Lemma 31 that $|\mathrm{Out}(\pi(U_j))\backslash\mathrm{Out}(\pi(U_{j-1}))| \leq |\mathrm{Out}(\pi(U_j^o))\backslash \mathrm{Out}(\pi(U_{j-1}^o))| \leq j-2$, proving (20) in this case.

We will treat the cases $i=\max\{i,j,k\}$ and $k=\max\{i,j,k\}$ at the same time. So assume $h=\max\{i,j,k\}$ where $h\in\{i,k\}$ and let h' be the member of $\{i,k\}$ other than h. By (22) and (23) we can apply Lemma 30 to $\{\tilde{a},\tilde{b},\tilde{c}\}=\{\pi(\tilde{m}_i),\pi(\tilde{m}_j),\pi(\tilde{m}_k)\}$. Hence there is no point in $\mathrm{Out}(\pi(U_h))$ adjacent to both $C(\pi(\tilde{m}_h))$ and $C(\pi(\tilde{m}_{h'}))$, so if

$$U_h^o=\bigcup_{l=1,\,l\neq h'}^{h} C(\tilde{m}_l) \quad\text{and}\quad U_{h-1}^o=\bigcup_{l=1,\,l\neq h'}^{h-1} C(\tilde{m}_l),$$

then we get as before by Lemma 31 that $|\mathrm{Out}(\pi(U_h))\backslash\mathrm{Out}(\pi(U_{h-1}))|\leq|\mathrm{Out}(\pi(U_h^o))\backslash \mathrm{Out}(\pi(U_{h-1}^o))|\leq h-2$, proving (20). Hence we have the lemma. □

For our final conclusion, we also need the following "folkloric" lemma:

Lemma 32. *A sequence of length p of real numbers contains a subsequence of length $\lceil\sqrt{p}\rceil$ which is either increasing or strictly decreasing.*

Using this we can prove

Lemma 33. *If $M\subseteq\mathbb{R}^3$ is a finite set with p elements, such that for some $i\in\{1,2,3\}$, $\pi_i(M)$ forms an antichain in $\mathbb{R}^2$, then there is a $j\in\{1,2,3\}$ such that $\pi_j(M)$ contains a chain of $\lceil\sqrt{p}\rceil$ elements in $\mathbb{R}^2$.*

Proof. By symmetry it suffices to show that if $\pi(M)$ forms an antichain in $\mathbb{R}^2$, then either $\pi_1(M)$ or $\pi_2(M)$ contains a chain of length $\lceil\sqrt{p}\rceil$ in $\mathbb{R}^2$. Assuming $\pi(M)$ forms an antichain, we can then label the elements of M as $\{(a_l,b_l,c_l)\in\mathbb{R}^3\colon l=1,2,\ldots,p\}$ where

$$a_1<a_2<\cdots<a_p, \tag{24}$$

$$b_1>b_2>\cdots>b_p. \tag{25}$$

Consider the finite sequence $\tilde{c}=(c_1,\ldots,c_p)$ of real numbers. By Lemma 32 $\tilde{c}$ contains a subsequence of length $\lceil\sqrt{p}\rceil$ which is either increasing or strictly decreasing.

If this subsequence of $\tilde{c}$ is increasing, then, by (24), $\pi_2(M)$ contains a chain in $\mathbb{R}^2$ consisting of $\lceil\sqrt{p}\rceil$ elements.

If this subsequence of $\tilde{c}$ is strictly decreasing, then by (25) $\pi_1(M)$ contains a chain in $\mathbb{R}^2$ consisting of $\lceil\sqrt{p}\rceil$ elements, which now completes the lemma. □

By this we can finally get what we need to determine $c_4(13)$:

Lemma 34. *$c_4(13)$ is not equal to 64.*

Proof. Assume there is a filter $U\in\mathscr{C}_4^{13}$ such that $|\mathrm{Out}(U)|=64$. If U is generated by $\tilde{m}_1,\ldots,\tilde{m}_{13}\in\mathbb{N}_0^4$ we may assume by Lemma 20 that $[\tilde{m}_1|\cdots|\tilde{m}_{13}]=[D|10\cdot I_{4\times 4}]$ where D is a 13×9 matrix each row of which consists of the positive integers $1,2,\ldots,9$.

Since $64=(13^2-3\cdot 13-2)/2$, Lemma 29 applies to $\{\pi_{34}(\tilde{m}_i)\colon i=1,2,\ldots,9\}\subseteq\mathbb{N}_0^2$, which therefore cannot contain any 3 element chain in $\mathbb{N}_0^2$ and must therefore contain an antichain of 5 elements. Let $M \subseteq \{\tilde{m}_1, \ldots, \tilde{m}_9\}$ be the 5 element set such that $\pi_{34}(M)$ is the antichain mentioned. Applying Lemma 33 to $\pi(M)$, we have that either $\pi_{41}(M)$ or $\pi_{42}(M)$ must contain a chain consisting of $\lceil\sqrt{5}\rceil = 3$ elements in $\mathbb{N}_0^2$, which again contradicts the second statement of Lemma 29. Our assumption that $|\mathrm{Out}(U)| = 64$ is therefore wrong and we have proved the lemma. $\square$

We now have the following theorem that summarizes what we have so far.

Theorem 35. *For $p \geq 4$ we have*

$$c_4(p) = (p^2 - 3p - 2)/2 \quad \textit{for } p \in \{4, 5, \ldots, 12\},$$

$$c_4(13) = 63,$$

$$c_4(p) \leq (p^2 - 3p - 4)/2 \quad \textit{for } p \geq 13.$$

Proof. By Lemma 27 we have the first statement. Lemmas 28 and 34 yield the second statement. Since $c_4(13) < (13^2 - 3 \cdot 13 - 2)/2$, we have by Corollary 26 the third and last statement of the theorem. $\square$

Remark. One sees clearly from the above theorem that $c_4(p)$ for $p \geq 4$ is not a polynomial in p, which by Theorem 19 could only have been of degree 2.

Acknowledgements

The author would like to thank Prof. Bernd Sturmfels for his keen interest and for his valuable help in getting this paper into present state.

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